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R & D NOTES

Developing Mass Transfer Boundary Layer with Power Law Velocity Profiles and Wall Boundary Conditions

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Recent work by the authors on the condensation of sulfuric acid from flue gases [Verhoff and Choi (1979)] focused attention on the problem of boundary layer mass transfer with varying wall concentration. The influence of varying wall concentration was mathematically simplified by assuming a power law dependency. The analysis also permitted the use of a power law velocity profile. It is the goal of this note to present a general derivation of the integral boundary layer analysis for developing mass transfer with a power law wall concentration function and a developed power law velocity profile.

Power law temperature boundary conditions have been used in thermal boundary layer analysis over the last thirty years [Chapman and Rubesin (1949), Levy (1952), Sparrow and Gregg (1958), and Nagendra and Tirunarayanan (1970)]. Similarly, power law velocity profiles have been used for heat and mass transfer boundary layer analysis [e.g. Mixon and Carberry (1960) and Levy (1952)].

Mixon and Carberry (1960) have shown that the power dependency of the Nusselt (Sherwood) Number on the Prandtl (Schmidt) Number depends upon the power law velocity function. Levy (1952) has shown that the power dependency of the Nusselt Number on the Prandtl Number depends upon both the power law velocity and the power law wall temperature boundary condition. However, since Levy's analysis was numerical he could not ascertain the exact dependency.

Theory

Herein, the problem of interest is the formation of the mass transfer boundary layer in a steady flowing stream such as may occur when the temperature of the surface drops below a condensation temperature or as may occur because of a change in surface concentration. The velocity distribution is assumed to be well established such that

$$v_y = 0 \tag{1}$$

$$\frac{\delta v_x}{\delta x} = 0 \text{ and } v_x = v_x(y) \tag{2}$$

Here, v_y represents the velocity in the y direction which is perpendicular to the flat plate and v_x represents the velocity in direction x which is parallel to the plate. This velocity function $v_x(y)$ is presumed known.

A mass balance on an incremental distance in the x direction yields the following equation

$$v_x \frac{\partial C}{\delta x} = D \frac{\delta^2 C}{\delta y^2} \tag{3}$$

The boundary conditions associated with this equation are as follows

$$@\ x = 0 \qquad C = C_\infty \tag{4}$$

$$y = \infty \qquad C = C_\infty \tag{5}$$

$$y = 0 \qquad C = C_o(x) \tag{6}$$

This problem could be solved by any of several techniques such as numerical integration of the differential equations, superposition of constant boundary condition equations, similarly

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transformation, and possible other techniques. All of these techniques are complicated mathematically compared to the integral boundary layer approach used herein.

Equation 3 can be integrated over y to obtain the following equation using the property shown in Eq. 2. (see Bird et al., 1960).

$$\frac{d}{dx}[\delta V_c C_\infty] = -D\frac{\delta C}{\delta y}\bigg|_{y=0} \quad \text{and} \tag{7}$$

$$\delta_B = \int_0^\infty \frac{v_x}{V_c}\left(\frac{C - C_\infty}{C_\infty}\right)dy \tag{8}$$

In the Pohlhausen integral boundary layer method, the dimensionless concentration profile is assumed similar for the developing concentration boundary layer. The independent variable defined in the boundary layer is

$$\eta = \frac{y}{\delta(x)} \tag{9}$$

Thus $\eta = 0$ at the wall and $\eta = 1$ at the edge of the mass transfer boundary layer.

The dimensionless concentration is defined as follows

$$\Gamma = \frac{C - C_\infty}{C_\infty} \tag{10}$$

Because the concentration at the wall is changing with distance the assumed form for the dimensionless profile will be slightly different.

$$\Gamma(\eta, x) = g(x)h(\eta) \tag{11}$$

In the integral boundary layer technique, the choice of function is arbitrary; however, the choice usually involves a function of a single variable. Herein, a function of two variables must be chosen; it is chosen as a product because, as with other integral boundary layer analyses, the concentration profile is presumed similar in shape at all distances x. The function choices must satisfy the boundary conditions.

$$h(1) = 0 \rightarrow C = C_\infty \ @\ \eta = 1 \quad \text{and}$$

$$h(0)\, g(x) = \frac{C_o(x) - C_\infty}{C_\infty}$$

or without loss of generality, $h(0)$ is assumed to be 1.

$$g(x) = \frac{C_o(x)}{C_\infty} - 1 \tag{12}$$

Also the velocity profile will be made dimensionless with the characteristic velocity, v_c, i.e.

$$\frac{v_x}{V_c} = f(ky) = f(k\eta\delta_B) \tag{13}$$

where k is an arbitrary constant chosen to be the reciprocal of a characteristic length. Substituting Eqs. 13, 11, 9, and 8 into Eq. 7 yields the following eqs.

$$\frac{-Dh'(o)g(x)}{V_c\delta_B} = \frac{d}{dx}\int_o^1 \delta_B f(k\delta_B\eta)g(x)h(\eta)d\eta \tag{14}$$

This then is the fundamental equation describing the dependency of δ_B layer thickness, on distance x.

This equation cannot be solved unless the dimensionless velocity function, $f(ky)$, is specified. It can be solved numerically for any velocity function, but it can be solved analytically for the two most commonly used velocity functions, i.e. the power law and the logarithmic function. This results because the power law and logarithmic function separate η and δ in $f(k\eta\delta)$. The power law will be explored in detail here. The logarithmic function requires more approximations.

Substituting the following power law function

$$f(k\delta\eta) = C'(k\delta\eta)^p = C'k^p\delta^p\eta^p \tag{15}$$

into Eq. 14 yields

$$-\frac{Dh'(o)g(x)}{V_c\delta_B} = \frac{d}{dx}C'\delta_B^{p+1}g(x)k^p\int_o^1 \eta^p h(\eta)d\eta \tag{16}$$

Let

$$C'\int_o^1 \eta^p h(\eta)d\eta = C'' \tag{17}$$

$$x^* = k_1 x \qquad \delta^* = k_2\delta \tag{18}$$

where x^* and δ^* are dimensionless distance and boundary layer thickness, respectively. Here, k_1 and k_2 are the inverse of characteristic lengths, the equation is

$$\frac{d\delta^*}{dx^*} + \frac{g'(x)}{(p+1)g(x)}\delta^* = -\frac{Dk_2^{p+2}h'(0)\delta^{*-(p+1)}}{V_c k_1 k^p C''(p+1)} \tag{19}$$

This is a Bernoulli differential equation whose solution is

$$\delta^* = \left[g(x^*)^{-\left(\frac{p+2}{p+1}\right)}\int_o^{x^*} -\frac{Dk_2^{p+2}h'(0)\,(p+2)}{V_c k_1 k^p C''(p+1)}\, g(\epsilon)^{\frac{p+2}{p+1}}\, d\epsilon\right]^{\frac{1}{p+2}} \tag{20}$$

The dimensionless local mass transfer rate is defined below.

$$\gamma = -\frac{Dh'(0)g(x^*)k_2}{\delta^* V_c} \tag{21}$$

or

$$\gamma = -\frac{Dh'(0)g(x^*)^{\frac{p+2}{p+1}}\, k_2}{V_c\left(\frac{Dk_2^{(p+2)}h'(0)\,(p+2)}{V_c k_1 k^p C''(p+1)}\right)^{\frac{1}{p+2}}\left[\int_o^x g(\epsilon)^{\frac{p+2}{p+1}}\, d\epsilon\right]^{\frac{1}{p+2}}} \tag{22}$$

Now assume that the variation of the wall conditions are also a power law function.

$$g(x^*) = x^{*r} \tag{23}$$

then

$$\delta^* = \left[\frac{Dk_2^{p+2}h'(0)\,(p+2)x^{*\frac{(r+1)p+2r+1}{p+1}}}{V_c k_1 k^p C''[(r+1)p+2r+1]}\right]^{\frac{1}{p+2}} \tag{24}$$

Substituting into Eq. 21 yields the following:

$$\gamma = -\frac{Dh'(0)x^{*\frac{rp+2r-1}{p+2}}}{V_c\left[-\frac{Dh'(0)\,(p+2)}{V_c k_1 k^p C''[(r+1)p+2r+1]}\right]^{\frac{1}{p+2}}} \tag{25}$$

Thus the dimensionless equation would be

$$\gamma = -\frac{-h'(0)x^{*\frac{rp+2r-1}{p+2}}}{\left[\frac{h'(0)\,(p+2)}{C''[(r+\ )p+2r+1]}\right]^{\frac{1}{p+2}}} \tag{26}$$

It can be seen that Eq. 21 is made dimensionless by defining

$$k_2 = \frac{V_c}{D} \tag{27}$$

and that Equation 25 becomes dimensionless using

$$k_1 = \frac{\left(\frac{V_c}{D}\right)^{p+1}}{k^p} \tag{28}$$

In summary then, Eq. 24 defines the dimensionless boundary layer thickness and Eq. 26 defines the dimensionless local mass transfer rate when the velocity profile is described by a power law and the concentration at the surface varies according to a power rule. These equations will be applied to two special cases: the mass transfer boundary layer is restricted to the laminar sublayer and the boundary layer extends into the turbulent core.

Laminar Sublayer

For the laminar sublayer, the characteristic velocity is the friction velocity, i.e. $V_c = V^*$. The velocity profile is defined as follows

$$\frac{V_x}{V^*} = \left(\frac{V^*}{\nu}\right)^p y^p \tag{29}$$

If the flow is laminar, p equals 1. However, the general power of p will be retained. Thus $k = V^*/\nu$ for this problem.

Neither Eq. 25 nor 26 is useful in their present form and they must be converted to the usual mass transfer coefficient depending on Schmidt Number and dimensionless distance.

In order to define the mass transfer coefficient, some concentration difference must be defined. Since the concentration along the wall varies (whereas most previous investigations had constant wall concentration), the concentration difference is somewhat arbitrary but will be chosen as follows.

$$\Delta C = 2C_\infty - C_\infty = C_\infty$$

This choice yields simplified solutions based upon the definition of the dimensionless concentration, Eq. 10. This is equivalent to a step change in the wall concentration from C_∞ to $2C_\infty$ at $X = 0$ if $r = 0$.

$$x^* = \left(\frac{V^*}{D}\right)^{p+1} \left(\frac{\nu}{V^*}\right)^p x = \left(\frac{V^*}{D}\right) \left(\frac{\nu}{D}\right)^p x \tag{30}$$

Substituting into Eq. 26 gives

$$\gamma = \frac{K}{V^*} = \frac{-h'(0)\left[\left(\frac{V^*}{D}\right)\left(\frac{\nu}{D}\right)^p x\right]^{\frac{rp+2r-1}{p+2}}}{\left[\frac{-h'(0)\,(p+2)}{C''(rp+2r+p+1)}\right)^{\frac{1}{p+2}}} \tag{31}$$

or this yields

$$N_{St} = C_L(N_{Sc})^{\frac{(p+1)\,(rp+2r-1)}{p+2}} (x^+)^{\frac{rp+2r-1}{p+2}} \tag{32}$$

where

$$C_L = \frac{-h'(0)}{\left[\frac{-h'(0)\,(p+2)}{C''(rp+2r+p+1)}\right]^{\frac{1}{p+2}}} \tag{33}$$

$$N_{Sc} = \left(\frac{\nu}{D}\right) \tag{34}$$

$$x^+ = \frac{xV^*}{\nu} \tag{35}$$

Equation 32 can be compared with prior theoretical analysis. If the laminar sublayer is considered to be truly laminar then the velocity profile is linear, i.e., $p = 1$, and the concentration change is truly a step change, i.e., $r = 0$, then the Schmidt number dependency is in fact $-2/3$ as has been presented previously (e.g. Son and Hanratty (1967)). Also the dependency upon dimensionless distance from the start of the step change is the $-1/3$ power as has been predicted in the past.

Also if the velocity profile is expanded in a Taylor series with the second power dominating, i.e. $p = 2$, with $r = 0$, the Schmidt No. dependency is $-3/4$ as has been predicted previously (see Shaw and Hanratty (1977)). The dependency on distance then would be a $-1/4$. If one subjectively considers the convective turbulent transport of mass (or momentum) near the wall, a velocity profile with a power less than 1 would be anticipated. But $p < 1$ and $r = 0$ (Eq. 32) indicates Schmidt No. dependency of greater than $-2/3$ up to $-1/2$ for a flat velocity profile ($p = 0$). However, no experimental data suggest a power of the Schmidt number greater than $-2/3$.

The experimental data of Shaw and Hanratty (1977) gives a steady state Schmidt number dependency of -0.704 which is between $-2/3$ and $-3/4$ but this does not correspond to a velocity power function of less than 1. Another explanation for this -0.704 power might be a non constant concentration boundary condition i.e. $r \neq 0$. Thus if one assumes that the velocity profile in the laminar sublayer is linear, i.e. $p = 1$, then the value of r which gives a Schmidt Number dependency of -0.704 is $r = -.018$. Except for the initial distance, $r = -.018$ corresponds to a very slightly decreasing wall concentration downstream. For this case at $x^* = 1$ the dimensionless wall concentration would be 1, and at $x^* = 1000$, the wall concentration would be 0.88. This could be accounted for with a slight kinetic limitation as the boundary layer developed. Also it could be accounted for by a slight roughness in the tube which was continuously altering the effective concentration boundary condition at the wall.

Because x^* involves the Schmidt Number, Eq. 23 implies a certain dependence of the boundary condition on Schmidt Number, which could exist in reality. It is possible to carry out the analysis for a fixed wall concentration profile (i.e. independent of Schmidt No.) and the results again indicate a Schmidt No. power dependency which is a function of r. Again, for a -0.704 power the value of r must be negative. It should be noted that in spite of the infinite concentration at $x = 0$, (see Eq. 23 for r negative and $x^* = 0$) the boundary layer thickness is zero at this point and thus all the boundary conditions are satisfied.

Turbulent Case

For the case of the mass transfer boundary extending into the turbulent core, the following velocity profile is of interest.

$$\frac{v_x}{V_\infty} = \left(\frac{y}{R}\right)^p = \left(\frac{1}{R}\right)^p y^p \tag{36}$$

Here R is the radius of the tube and p would be a fractional power (e.g. 1/7). Thus for this case the following dimensionless variables are defined.

$$\gamma = \frac{K}{V_\infty},\ x^* = \left(\frac{V_\infty}{D}\right)^{p+1} (R)^p x$$
$$= \left(\frac{2eV_\infty R}{\mu}\right)^{p+1} \left(\frac{\nu}{D}\right)^{p+1} \left(\frac{1}{2}\right)^p \frac{x}{2r} \tag{37}$$

These dimensionless numbers then can be substituted into Eq. 26 to obtain the Stanton No. as a function of Reynolds No., Schmidt No. and dimensionless distance.

$$N_{St} = C_T(N_{Re}N_{Sc})^{\frac{(p+1)\,(rp+2r-1)}{p+2}} (x^+)^{\frac{rp+2r-1}{p+2}} \tag{38}$$

where

$$C_T = C_L[2]^{\frac{-p(rp+2r-1)}{p+2}}$$

$$N_{Re} = \frac{2V_\infty R}{\nu}$$

$$N_{Sc} = \left(\frac{\nu}{D}\right)$$

$$x^+ = \frac{x}{2R}$$

This equation has little meaning for mass transfer since there are few fluids with very low Schmidt Numbers ($\simeq 0.001$) such that the boundary layer will penetrate the turbulent core in the entrance region. However, it is useful to note that this correlation predicts that the Stanton Number should depend upon the product of the Reynolds Number and the Schmidt Number. This level of dependency was found for the analogous heat transfer problem with liquid metals [see Bird, et al. (1960)]. For liquid metals no simple power law rule as suggested by Eq. 38 could be found.

In summary, a simple integral boundary layer analysis suggests various dependencies for the Stanton number as a function of Schmidt Number, Reynolds Number, and dimensionless entry length for the case of power law velocity profiles and wall boundary concentrations.

NOTATION

C_T, C_L, C', C'' = constants
C = concentration
C_o = concentration at transfer surface
C_∞ = concentration far from transfer surface
D = diffusivity
f = dimensionless velocity profile
g = dimensionless concentration function of x
h = dimensionless concentration function of y
k, k_1, k_2 = constants
K = local mass transfer coefficient
N_{Re} = Reynolds Number
N_{Sc} = Schmidt Number
N_{St} = Stanton Number
p = power of velocity power law
r = power law for wall concentration dependency
R = radius of tube
v_x, v_y = x and y components of fluid velocity
V_c = characteristic velocity
V^* = friction velocity
x = coordinate in direction of flow
x^+, x^* = dimensionless distance in direction of flow
y = coordinate perpendicular to flow
y^* = dimensionless distance perpendicular to flow

Greek Symbols

γ = dimensionless local mass transfer rate (Stanton number)
Γ = dimensionless concentration
δ_B = boundary layer thickness
δ^* = dimensionless boundary layer thickness
η = similarity variable
ν = fluid kinematic viscosity
ϵ = integration variable

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On the Possible Existence of Concentration Stability Effects within a Continuous Well-Mixed Reactor When a Solid Catalyzed Liquid to Gas Reaction Occurs

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During recent years, considerable effort has been devoted to the examination of temperature stability effects within chemical reactors (see, for example, Aris, 1975). The field's seminal paper appears to have been that of Van Heerden (1953). Concentration stability effects may also manifest themselves under isothermal operating conditions. For example, Matsuura and Kato (1967) have re-